Enthalpy-based feedback control algorithms for the Stefan problem

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Abstract— A control law is derived for full-state feedback control of the single-phase Stefan problem, used as a model of industrial casting processes. A conceptually novel controller design approach is proposed for this problem, with the control law chosen to ensure exponential stability of the average enthalpy, and proven to guarantee asymptotic convergence of both the temperature field and the solidification front position to a desired reference. A plausible output feedback control algorithm is also given that demonstrates good behavior in simulation.

I. INTRODUCTION

Casting processes present significant control challenges. Many of the key quality and operational goals for industrial casting processes can be met by matching a desired temperature history while the material cools. For example, in continuous casting of steel slabs, a common defect is cracking, either internal or on the surface. Transverse surface cracks can be caused by mechanical strain while the stillsolidifying slab is bent to pass through the casting machine. Since ductility of steel during the cooling process is strongly temperature-dependent, these cracks can be avoided by regulating the surface temperature. Internal cracks, in contrast, are dependent on the temperature history at the solidification front. An important safety goal in continuous casting is the prevention of "whales." The molten steel inside the slab exerts a ferrostatic pressure outward on the shell. If the steel is not fully solidified by the time it leaves the casting apparatus, this pressure causes the shell to bulge outward, creating a defect that is called a "whale" due to resemblance to the latter to the shape the bulging makes. At a minimum, this damages the equipment and causes a long work stoppage while the plant must wait for the whale to fully cool before it can be cut out and removed. In the worst case, liquid steel can escape through the shell, potentially causing more severe damage or serious injuries. In terms of the temperature profile, whale prevention is equivalent to ensuring the temperature at caster exit is below the solidification temperature.

Hence, achieving the quality and safety goals simultaneously requires regulating the entire distributed temperature profile of the casting. This task, however, is very challenging – the solidification process is nonlinear in a manner that existing distributed parameter control algorithms are not suited to handle.

The Stefan problem is the name for a type of nonlinear partial differential equation (PDE) that is commonly used to model such phase changes in pure materials. In such a PDE, the domain is divided into subdomains, one for each phase, separated by a moving boundary. Within each subdomain, the temperature evolves according to the usual diffusion equation. At the solid-liquid interface, the energy balance is adjusted to account for the phase change. This leads to a differential equation for the evolution of the interface position in terms of the heat flux or temperature gradient on either side, which in general is not continuous at the interface. This problem cannot be reduced to a semilinear PDE, or a bounded perturbation of a linear PDE. For this reason, finding a control algorithm has proven difficult.

There have been many algorithms suggested, but all have weaknesses for practical implementation. The numerical optimization methods in [1, 2] can handle realistic metallurgical constraints, but since the calculations are complex and non-linear, they cannot realistically run in realtime. As such, they are suited mostly for open loop control. Other researchers [3-5] have attempted to solve the inverse Stefan problem. Since these approaches focus on the interface only, they could be used for whale prevention, but not for crack prevention. Finally, several attempts have been made in recent years to provide feedback control, which is naturally more suitable for real-time implementation. The methods in [6, 7] apply PI control, in [6] to the surface temperature and in [7] to the solidification front. By applying the finite-dimensional approach, both approaches achieve good results for one of the key process goals, but not all of them. The approaches in [8, 9] are more sophisticated, but assumes unrealistic "thermostat" style actuation.

Finally, in [10], a full-state feedback control law was designed to have the temperature converge to a reference temperature profile. While guaranteeing uniform asymptotic stability of the error, the derived control law could potentially require unbounded actuation. This caused the closed-loop system to behave erratically in simulations. To ensure well-posedness of the problem, the control law needed to be saturated, which was not considered in the convergence proof. Moreover, even the unbounded control laws given could not be proven to guarantee convergence of the interface position to the reference.

We claim that the fundamental issue with the approach in the latter work was the use of a *temperature-based* Lyapunov functional. In solidification, the important quantity is the *enthalpy* of the material, which changes sharply and nonlinearly with temperature. This novel conceptual framework is more clearly connected to the physics of the problem, and the effect of the available control input. This allows for a control algorithm that guarantees convergence of the full state, temperature and interface location naturally, without requiring unrealistic actuation.

In this paper, as in [10], we attempt to regulate the temperature of a solution to the Stefan problem relative to a

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reference solution. This reference solution will typically be designed offline by a metallurgist, e.g. using the numerical optimization techniques in [1, 2], to achieve the desired quality goals from a nominal initial condition. The control objective, then, is to asymptotically converge to this desired trajectory from a different initial condition.

In Section II, we describe the problem setting. In Section III, we give the main result of the paper, a control law that guarantees simultaneous asymptotic convergence of the temperature and interface position to the reference profile. In Section IV, we briefly discuss some potential extensions of this new framework.

II. THE STEFAN PROBLEM

A. The Two-Phase Stefan Problem

Following the modeling approach of [11], the temperature of steel in a continuous slab caster can be modeled accurately using a one-dimensional domain. Slab casters are so named because they are much wider (larger than 1 m, usually) than they are thick (on the order of 0.1 m). Away from the narrow faces, heat transfer in the width direction is negligible. In the "axial" or casting direction, heat is advected by the material moving through the caster at the casting speed as well as conducted. A scaling argument can be made that advection dominates conduction in this direction. Then, by using a frame of reference that moves with the material, i.e. the Lagrangian approach, axial heat transfer can be neglected. This leaves a one-dimensional transverse slice of material traveling through the caster at the casting speed as the spatial domain of the system. We further assume that the temperature is symmetric across the centerline of this slice.

We denote the temperature in the material as T(x,t) on the domain $(x,t) \in (0,L) \times [0,\infty)$ where x = 0 and x = Lcorrespond to the surface and center of the slab, respectively. We denote the position of the liquid-solid interface as s(t). On this domain, the two-phase Stefan problem is:

$$T_{t}(x,t) = aT_{xx}(x,t), \qquad x \in (0,L) - \{s(t)\},$$

$$T(s(t),t) = T_{f}, \qquad (1)$$

$$T_{x}(0,t) = u(t), \qquad T_{x}(L,t) = 0,$$

$$T(x,0) = T_{0}(x), \qquad \dot{s}(t) = -bT_{x}(x,t)|_{x=s(t)^{-}}^{x=s(t)^{+}}, \qquad (2)$$

$$s(0) = s_{0}$$

The parameters a, b, and L are all positive.

The physical meaning of these equations is that the material is solid between the surface and the moving boundary s(t), and liquid between the boundary and the centerline. Inside each phase, the temperature evolves according the usual heat diffusion equation. The Stefan condition (2) follows from a heat balance at the interface, taking into account the energy required to freeze a differential portion of the material. In these equations, T_f is the melting

temperature. The thermal diffusivity is $a = k / \rho c_p$, where k is the thermal conductivity, ρ the density, and c_p the specific heat. The coefficient in the Stefan condition (2) is $b = k / \rho L_f$, where L_f is the latent heat of fusion. The Neumann boundary control input u(t) is directly proportional to the heat flux at the surface from the cooling water sprays.

B. The Single-Phase Stefan Problem

In order for the initial condition to be physically sensible, s_0 should be in (0,L), and $T_0(x)$ should be less than T_f for $x < s_0$ and greater than T_f for $x > s_0$. That is, the initial liquid portion should be above the melting temperature and the initial solid portion should be below. However, in actual casting conditions, the temperature in the liquid is negligible. A typical superheat (the difference between the initial temperature in the liquid and the melting temperature) is only around 25°C. In comparison, the average surface temperature of the steel in the caster is around 500°C less than the melting temperature. Moreover, due to fluid flow in the liquid, the temperature in the liquid reaches steady state much more quickly than conduction alone would achieve. Hence, a common modeling assumption for this problem is to assume the initial temperature in the liquid is uniformly equal to the melting temperature, T_{f} .

This reduces the dynamics of the problem to only those of the solid phase. However, for ease of notation and calculation, we will continue to use the PDE (1)-(2) for the rest of this paper, with the following assumption:

(A1) The initial conditions satisfy: $0 < s_0 < L$, $T_0(x) < T_f$ and is non-decreasing for all $0 \le x < s_0$, and $T_0(x) = T_f$ for all $x \ge s_0$, and are piece-wise smooth.

We also make an assumption on the control input u(t):

(A2) inf $u(t) \ge 0$ and $\sup u(t) < \infty$.

Finally, we make one assumption on the temperature profile:

(A3) During the time in question, 0 < s(t) < L.

The first two assumptions are physically sensible for the target application. Castings solidify from the outside in, and since they are at high temperatures there is always some minimum cooling at the surface due to radiation and natural convection. The third assumption is meant to ensure the problem remains nonlinear. After solidification ends, or before it begins, the problem is linear and may be dealt with using any of the available well-studied methods that work for such problems (e.g. [12]). Whenever possible, we will give conditions that ensure the assumptions are satisfied.

C. Reference temperature and error

As stated above, the control objective is to match some ideal temperature profile. We denote this profile as $\overline{T}(x,t)$

and $\overline{s}(t)$. We assume it is defined as the solution to (1)-(2) under some known initial conditions $\overline{T}_0(x)$ and \overline{s}_0 , and control $\overline{u}(t)$.

We use the notation $\tilde{T}(x,t) \coloneqq T(x,t) - \overline{T}(x,t)$ and $\tilde{s}(t) = s(t) - \overline{s}(t)$ for, respectively, the temperature and interface position reference errors. We also denote $\tilde{u}(t) = u(t) - \overline{u}(t)$. Since solutions to (1)-(2) are continuous everywhere on the spatial domain, and at least twice spatially differentiable everywhere except at the moving boundary, the reference error \tilde{T} inherits these properties. In particular, the temperature error satisfies the equations,

$$\begin{split} \tilde{T}_{t}(x,t) &= a\tilde{T}_{xx}(x,t), \qquad x \in (0,L) - \left\{s(t)\right\} \cup \left\{\overline{s}(t)\right\}, \\ \tilde{T}_{x}(0,t) &= \tilde{u}(t), \quad \tilde{T}_{x}(L,t) = 0. \end{split}$$
(3)

Moreover, if $s(t) \neq \overline{s}(t)$, then $\overline{T}_x(s(t)^-, t) = \overline{T}_x(s(t)^+, t)$, which means

$$\dot{s}(t) = -b \,\tilde{T}_{x}(x,t) \Big|_{x=s(t)^{+}}^{x=s(t)^{+}},\tag{4}$$

and, similarly,

$$\dot{\overline{s}}(t) = b \, \widetilde{T}_x(x,t) \Big|_{x=\overline{s}(t)^+}^{x=\overline{s}(t)^+} \,.$$
(5)

As they are continuous and piecewise continuously differentiable, T, \overline{T} , and \tilde{T} are all in the Sobolev space $H^1(0,L)$ at all times. We will denote, when needed, $s_1(t) = \min\{s(t), \overline{s}(t)\}$, and $s_2(t) = \max\{s(t), \overline{s}(t)\}$. From this point onwards we will simplify notation by dropping the argument(s) for most terms, and using for example, $\tilde{T}(x)$ instead of $\tilde{T}(x,t)$.

D. Open loop simulation

Table 1 gives the parameters used for all simulations in this paper. The properties approximate ultra-low carbon (ULC) steel. The initial temperatures used are shown in Figure 1. Simulations were performed using an enthalpybased code, rather than a true numerical implementation of the moving boundary. However, the simulation code has matches well with a known analytical solution to the Stefan problem from [13].

Symbol	Description	Value
Α	thermal diffusivity	$2.27 \text{ x } 10^{-5} \text{ W/m} \cdot \text{K}$
В	Stefan condition constant	4.13 x 10 ⁻⁸ W/m·K
T_{f}	melting temperature	1811 K
L	half-thickness of material	0.2 m
\overline{u}	constant reference input	3000 K/m

TABLE 1. THERMODYNAMIC PROPERTIES USED IN SIMULATIONS

Figures 2 and 3 show, respectively, the temperature error and interface position when open-loop control is used. That is, there is an initial error, but u is never changed from the



Figure 1. Initial conditions for simulations



Figure 2 Reference temperature error using nominal control input



Figure 3. Interface position error using nominal control input

constant reference input. A more detailed discussion of these figures is given in Section IV.A, below.

III. FULL-STATE FEEDBACK

A. Preliminaries and notation

If the solution (T, s) to the PDE (1)-(2) satisfies assumptions (A1)-(A3), we can immediately conclude a few facts. First, $T_0(x) < T_f$ and is non-decreasing for all $0 \le x < s(t)$, which in turn implies $\dot{s}(t) \ge 0$ for all $t \ge 0$. For the single-phase Stefan problem, $\tilde{T}(x) \equiv 0$ when $s_2 \le x \le L$.

Since (1) is parabolic on the subdomains, if (A2) holds then T_x is uniformly bounded by a constant depending on the initial condition and bounds on u. (See, e.g., Theorem 11.1 from Section III.11, p. 211 of [14].) By (2), this means that the solidification front speed is bounded, i.e.

$$0 \le s_{\min} \le \dot{s} \le s_{\max} < \infty . \tag{6}$$

Also, as a consequence of the version of Poincare's inequality given in [12] (Lemma 2.1, p. 17), we have an estimate on $||T||_2$:

$$\int_{0}^{L} T^{2} dx \leq 2T^{2} \left(s \right) + 4L^{2} \int_{0}^{L} T_{x}^{2} dx = 2T_{f}^{2} + 4L^{2} \int_{0}^{L} T_{x}^{2} dx.$$
(7)

That is, both T and T_x are bounded in the $L_2(0,L)$ norm, and hence T is bounded in the Sobolev space $H^1(0,L)$. Similarly, Agmon's Inequality (Lemma 2.4, p. 20, *ibid*) ensures that |T| is also uniformly bounded.

We now introduce the notation:

$$h(T) := \begin{cases} \frac{1}{a}T, & \text{if } T < T_f \\ \frac{1}{a}T + \frac{1}{b}, & \text{if } T \ge T_f \end{cases}.$$

In the physical system, this is proportional to the enthalpy of the material at the given temperature. We will use the notation $\tilde{h} := h(T) - h(\overline{T})$ for the difference in enthalpy and

$$\tilde{H} := \int_0^L \tilde{h} dx = \frac{1}{a} \int_0^L \tilde{T} dx - \frac{1}{b} \tilde{s}, \qquad (8)$$

for the total difference. Taking the time derivative of this value, and using the fact that \tilde{T} is continuous,

$$\frac{d}{dt}\tilde{H} = \frac{d}{dt} \left(\frac{1}{a} \int_0^L \tilde{T} dx - \frac{1}{b} \tilde{s} \right)$$
$$= \frac{1}{a} \int_0^L T_t dx - \frac{1}{b} \dot{\tilde{s}}$$

Using equations (3)-(5),

$$\frac{d}{dt}\tilde{H} = \int_{0}^{L}\tilde{T}_{xx} - \frac{1}{b}\dot{s} + \frac{1}{b}\dot{s}
= \tilde{T}_{x}\Big|_{0}^{s_{1}^{-}} + \tilde{T}_{x}\Big|_{s_{1}^{+}}^{s_{2}^{-}} + \tilde{T}_{x}\Big|_{s_{-}^{-}}^{L} + \tilde{T}_{x}\Big|_{s_{-}^{-}}^{s^{+}} + \tilde{T}_{x}\Big|_{s_{-}^{-}}^{\overline{s}^{+}}$$

$$= -\tilde{T}_{x}\left(0\right) + \tilde{T}_{x}\left(L\right) = -\tilde{u}.$$
(9)

B. Control law

With these estimates in mind, we now state the main result of this paper:

Theorem 1. Suppose the initial conditions satisfy assumption (A1), the reference temperature profile satisfies assumptions

(A2) and (A3), the boundary condition satisfies the control law

$$u(t) = \overline{u}(t) + k\widetilde{H}(t), \qquad (10)$$

and the closed-loop system satisfies assumptions (A2) and (A3).

Then the reference temperature error \tilde{T} converges asymptotically to 0 uniformly over the domain, and the interface position error \tilde{s} converges to 0 asymptotically as well.

Proof: In light of (9), if the control law (10) is used, $|\tilde{H}|$ and $|\tilde{u}|$ are exponentially decreasing. As noted above, if all assumptions are satisfied, T and \overline{T} , and consequently also \tilde{T} are bounded in $H^1(0, L)$ over time. Then, by the definition of \tilde{H} in (8), $|\tilde{s}|$ must also be bounded.

Similar to the main proof in [10], we apply an infinitedimensional invariance principle from [15]. Consider the Lyapunov functional candidate

$$V(\tilde{T}) \coloneqq \frac{1}{2} \int_0^L \tilde{T}^2 dx - \frac{a}{b} T_f(s + \overline{s}) + 2\frac{a}{b} T_f L$$

on the state space of the error system, $(\tilde{T}, \tilde{s}) \in H^1(0, L) \times \mathbb{R}$. This function is clearly continuous on that space, and non-negative on trajectories of the system.

Taking the time derivative of the first term,

$$\frac{d}{dt}\frac{1}{2}\int_{0}^{L}\tilde{T}^{2}dx = \frac{1}{2}\frac{d}{dt}\left[\int_{0}^{\overline{s}}\tilde{T}^{2}dx + \int_{\overline{s}}^{s}\tilde{T}^{2}dx + \int_{s}^{L}\tilde{T}^{2}dx\right]$$
(11)
$$= \int_{0}^{\overline{s}}\tilde{T}\tilde{T}_{t}dx + \int_{\overline{s}}^{s}\tilde{T}\tilde{T}_{t}dx + \int_{\overline{s}}^{L}\tilde{T}\tilde{T}_{t}dx.$$

Then, applying the PDE (3), and integrating by parts,

$$\frac{d}{dt} \frac{1}{2} \int_{0}^{L} \tilde{T}^{2} dx = a \int_{0}^{s_{1}} \tilde{T} \tilde{T}_{xx} dx + a \int_{s_{1}}^{s_{2}} \tilde{T} \tilde{T}_{xx} dx + a \int_{s_{2}}^{L} \tilde{T} \tilde{T}_{xx} dx$$

$$= a \tilde{T} \tilde{T}_{x} \Big|_{0}^{s_{1}} - a \int_{0}^{s_{1}} \tilde{T}_{x}^{2} dx + a \tilde{T} \tilde{T}_{x} \Big|_{s_{1}^{+}}^{s_{2}^{-}} - a \int_{s_{1}}^{s_{2}} \tilde{T}_{x}^{2} dx$$

$$+ a \tilde{T} \tilde{T}_{x} \Big|_{s_{2}^{+}}^{L} - a \int_{s_{2}}^{L} \tilde{T}_{x}^{2} dx$$

$$= -a \tilde{T} (0) \tilde{T}_{x} (0) - a \tilde{T} \tilde{T}_{x} \Big|_{s_{2}^{-}}^{s_{2}^{+}} - a \tilde{T} \tilde{T}_{x} \Big|_{s_{-}^{-}}^{s_{-}^{+}}$$

$$+ a T (L) T_{x} (L) - a \int_{0}^{L} T_{x}^{2} dx.$$

Applying the boundary conditions from (1) and the relationships (4) and (5),

$$\frac{d}{dt}\frac{1}{2}\int_{0}^{L}\tilde{T}^{2}dx = -a\tilde{T}(0)\tilde{u} - a\int_{0}^{L}\tilde{T}_{x}^{2}dx + \frac{a}{b}\tilde{T}(s)\dot{s} - \frac{a}{b}\tilde{T}(\overline{s})\dot{\overline{s}}$$
(12)

Differentiating the second term using gives

$$\frac{d}{dt}\frac{a}{b}T_{f}\tilde{s} = \frac{a}{b}T_{f}\left(\dot{s} + \dot{s}\right)$$
(13)

Combining (12) and (13),

$$\frac{d}{dt}V(\tilde{T}) = -a\tilde{T}(0)\tilde{u} - a\int_{0}^{L}\tilde{T}_{x}^{2}dx$$

$$+\frac{a}{b}(\tilde{T}(s) - T_{f})\dot{s} - \frac{a}{b}(\tilde{T}(\bar{s}) + T_{f})\dot{\bar{s}} \qquad (14)$$

$$= -a\tilde{T}(0)\tilde{u} - a\int_{0}^{L}\tilde{T}_{x}^{2}dx - \frac{a}{b}(\bar{T}(s)\dot{s} + T(\bar{s})\dot{\bar{s}}).$$

We already noted that $|\tilde{u}|$ decreases exponentially, and

$$\left|\tilde{T}(0)\right| = \left|\tilde{T}(0) - \tilde{T}(L)\right| = \left|\int_{0}^{L} \tilde{T}_{x}\right| \le \sqrt{L} \left\|\tilde{T}_{x}\right\|_{2}$$
(15)

where the final inequality follows from that of Cauchy-Schwarz. Thus, the first term is exponentially decreasing. Under the assumptions, both \dot{s} and $\dot{\bar{s}}$ are positive and bounded below, as discussed above. Since the temperatures are bounded, choosing an appropriate temperature scale ensures $\overline{T}(s)$ and $T(\overline{s})$ are also non-negative. So, after enough time,

$$\frac{d}{dt}\tilde{V}(t) \le -a \int_0^L \tilde{T}_x^2 dx \tag{16}$$

Then, applying the Poincare inequality given in [12] (Lemma 2.1, p 2.30),

$$\begin{aligned} -a \int_{0}^{L} \tilde{T}_{x}^{2} dx &\leq -\frac{a}{4L^{2}} \int_{0}^{L} \tilde{T}^{2} dx + 2\tilde{T}^{2} \left(L \right) \\ &= -\frac{a}{4L^{2}} \int_{0}^{L} \tilde{T}^{2} dx := -W \left(\tilde{T} \right). \end{aligned}$$

We now apply the infinite dimensional invariance principle in [15], Theorem 6.3, p. 195. Using that notation, denote \mathcal{X} to be $H^1(0,L) \times \mathbb{R}$, the state space of the problem, and \mathcal{Y} to be $C^0(0,L) \times \mathbb{R}$. Denote \hat{W} and \hat{V} to be the extensions of W and V respectively to \mathcal{Y} . By an application of the Rellich-Kondrakov theorem (Theorem 5.5, p. 269 in [16]) and the Ascoli-Arzela criterion (Appendix C.7, p. 635, *ibid*), \mathcal{X} can be shown to be compactly embedded in \mathcal{Y} . As noted above, the trajectories of the error system are bounded in the space \mathcal{X} . All conditions of the theorem are met, and we conclude that all trajectories of the system converge to the set

$$\mathcal{M}_{3} \subset \left\{ y \in \mathcal{Y} : \hat{W}(y) = 0 \right\} = \left\{ \tilde{T} \equiv 0 \right\}$$

in the \mathcal{Y} -norm. That is, \tilde{T} converges to 0 uniformly.

Then, since both \tilde{T} and $\left|\tilde{H}\right|$ converge to 0, according to the definition (8), \tilde{s} must converge to 0 as well. \Box

C. Discussion and simulation

The main weakness of this result lies in the applicability of the assumptions. Assumption (A1) depends only on the initial conditions, and is entirely reasonable. Assumption (A2) can be satisfied by choosing the controller gain ksufficiently small. For example,

$$k < \frac{\inf \overline{u}}{\tilde{H}(t=0)}$$

will ensure that assumption (A2) holds on u. This gain can be further adjusted to limit the amount of control actuation, at the trade-off of reducing the rate of convergence.

Assumption (A3) is more problematic. Obviously, the material will eventually completely solidify. The bounds necessary to the proof can still be shown to hold, allowing s = L and neglecting the Stefan condition (2) in the PDE. The temperature condition, $T(s) = T_f$, which was used in deriving (7) and (15), will also be lost. However, it is fairly easy to find equivalent limits on the linear system. The conclusion, that the temperature error converges uniformly to 0, will still hold, but this is not a satisfying result. Hence, for certain initial errors and reference profiles, the temperature error may not converge very closely before solidification is completed. In general, as the speed \overline{s} of the reference solidification front speed gets smaller, the temperature and interface position errors will converge closer to 0 before the material is completely solidified.

A stronger result would be to guarantee the rate of convergence, but we are currently unable to prove this mathematically. We can, however, offer simulation evidence that this is the case. Figures 4-6 show a simulation of this control algorithm. The actual system (T,s) is given a different initial condition than the reference system $(\overline{T}, \overline{s})$. The specific initial conditions used are shown in Figure 1, and the rest of the simulation parameters are given in Table 1.

Due to the spatial discretization, the location of the interface s has some inherent uncertainty that can lead to numerical noise in control law (10). Since the simulation method used is based on the calculated enthalpy at each node, this was used directly in the control calculations instead. This ensures the control response, seen in Figure 6, does not spike whenever the estimated interface position passes a node.

In this simulation, the reference temperature and interface position errors, in Figures 4 and 5, respectively, are clearly converging exponentially fast. Work is continuing into proving such bounds on the rate of convergence.

Finally, we note that the proof was only specific to the single-phase Stefan problem in applying Agmon's and Poincare's Inequalities, e.g. in (15), which required that $\tilde{T}(L) = 0$, and in using the estimate (6) to ensure negative definiteness of the Lyapunov functional time derivative. The former is easily dealt with by using a weaker bound on $\|\tilde{T}\|_{\infty}$ based on compactly embedding $H^1(0,L)$ into $C^0(0,L)$, as described above. The inequality (6) is still usually true for the two-phase Stefan problem, but it is difficult to determine exact conditions on the inputs and initial conditions that ensure the condition holds. Currently, the best that can be proven for the two-phase Stefan problem is the following:

Corollary 1. Suppose the system satisfies the conditions of Theorem 1, with the exception that we allow $T_0(x) > T_f$ in the range $s_0 < x \le L$, and a similar condition on the reference. Then, if both $\dot{s} \ge 0$ and $\dot{\bar{s}} \ge 0$ for the entire time,

the reference temperature error converges to 0 in the uniform norm, and the interface position error converges to 0 as well.

IV. EXTENSIONS

A. Temperature-convergent control Returning to the calculation (14), if



Figure 4. Reference temperature error using state-feedback control law (10)



Figure 5. Solid-liquid interface position using state-feedback control law (10)



Figure 6. Boundary control input using state-feedback control law (10)

 $\tilde{u}(0)\tilde{T}(0) \ge 0, \qquad (17)$

then the estimate (16) still holds, and the invariance principle can be applied. However, the total enthalpy may not converge to 0, so temperature convergence does not imply convergence of the interface position. In fact, for an arbitrarily small temperature error, the interface position error can still be arbitrarily large. To summarize:

Theorem 2. Suppose the reference system and actual system satisfy assumptions (A1)-(A3) and condition (17). Then the reference temperature error converges uniformly to 0.

This result covers a class of controllers including proportional control based on the surface temperature and similar finite-dimensional approaches. However, there are great drawbacks to this simplistic approach, which are illustrated by the simulations shown in Figures 2 and 3.

Both actual and reference system in these simulations use the reference input for the boundary condition, i.e. $\tilde{u} \equiv 0$. Under Theorem 2, the temperature error should converge to 0. Figure 2 does show the temperature error converging, but extremely slowly. There is also the problem of the interface position. From (9), we can conclude that the total enthalpy error is constant over time. Then, given (8), the interface position error must converge to a constant that is non-zero if the initial total enthalpy error is non-zero. This behavior can be observed in Figure 3.

B. Estimation and output feedback

Unfortunately, full-state feedback is not realistic for this problem. Enthalpy cannot be directly measured, only temperature can. Moreover, only the steel surface temperature can realistically be measured during actual casting. For an implementable control algorithm, we need an estimation scheme.

One approach, as proposed in [10], is to use the result of Theorem 2 to provide an estimation algorithm. The idea is to use an estimate (\hat{T}, \hat{s}) of the actual system that is a solution to (1)-(2) with left-hand boundary condition

$$\hat{T}(0,t) = T(0,t)$$

Under Theorem 2, the temperature estimation error should converge to 0, but the interface position estimation may not converge to the true location.

In simulations, we have achieved better performance using an estimation procedure that adjusts the solidification front position as a function of the surface temperature estimation error. The method, which is not yet proven to work, is as follows:

Let \hat{T} and \hat{s} be an estimate of the temperature T and interface location s, respectively. Let the estimates satisfy the following PDE



Figure 8. Temperature estimation error under output injection method (19)-(20)



Figure 7. Reference temperature error under output-feedback control law (21)

$$\hat{T}_{t}(x,t) = a\hat{T}_{xx}(x,t), \qquad x \in (0,L) - \{\hat{s}(t)\},$$

$$\hat{T}(\hat{s}(t),t) = T_{f}, \qquad (18)$$

$$\hat{T}_{x}(0,t) = u(t), \quad \hat{T}_{x}(L,t) = 0,$$

$$\hat{T}(x,0) = \hat{T}_{0}(x),$$

$$\hat{s}(t) = -b(\hat{T}_{x}(\hat{s}(t)^{+}) - \hat{T}_{x}(\hat{s}(t)^{-})) + l(\hat{T}(0,t) - T(0,t)), \qquad (19)$$

$$\hat{s}(0) = \hat{s}_{0}$$

where the initial conditions satisfy assumption (A1) and the estimation gain l > 0.

ŝ

Conjecture. Suppose (T, s), $(\overline{T}, \overline{s})$, and (\hat{T}, \hat{s}) all satisfy the conditions for Theorem 1. Then under the control law

$$u = \overline{u} - k \left[\frac{1}{a} \int_0^L \left(\hat{T} - \overline{T} \right) dx - \frac{1}{b} \left(\hat{s} - \overline{s} \right) \right], \tag{20}$$

the temperature reference and estimation errors uniformly converge to 0, and the interface position reference and estimation errors converge to 0.

Figures 7-10 show a simulation using this outputfeedback control law. The initial estimation is the same as the reference, i.e.



Figure 10. Solid-liquid interface location under control law (21)



Figure 9. Boundary control action under output-feedback control law (21)

$$\hat{T}_0 = \overline{T}_0, \quad \hat{s}_0 = \overline{s}_0.$$

The estimation and reference errors appear to be converging to 0 exponentially. However, this apparent result remains conjecture, and is a subject of ongoing work.

V. CONCLUSION

The goal of this paper is to improve on previous results on control of solidification systems by changing the error) framework. The key insight is that when the controller design is based on a norm of the temperature error, the controller may attempt to over-compensate for the nonlinear dynamics, leading to unbounded, quickly varying actuation, as seen in [10]. This paper uses the novel conceptual framework of focusing instead on the enthalpy error for the control law. The control law devised in this formulation remains bounded, and is able to simultaneously provide asymptotic convergence for both the temperature and liquid-solid interface position of the desired reference.

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